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LOCAL-MAXIMUM-LIKELIHOOD ESTIMATION OF THE PARAMETERS OF THREE-PARAMETER LOGNORMAL POPULATIONS FROM COMPLETE AND CENSORED SAMPLES

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These corrections were supplied by the authors:

P. 842. Both authors are employed at Wright-Patterson Air Force Base, but the second author is with the Air Force Institute of Technology, not the Aerospace Research Laboratories.

P. 845. On the fourth line from the bottom, the lower limit of the integral should be $\hat{z}_{r+1}$.

P. 846. On the last line of equation (3.16), the sign preceding $\hat{z}_{r+1}f(\hat{z}_{r+1})/q_1$ should be positive.

P. 847. In Table 1, the entries in the second column from the right, first line, and the third column from the right, third and fifth lines, should be −0.111959, −0.352750, and −0.513932, respectively.

P. 850. In Table 2, the last two column headings in the first section of the lower half of the table should be Cov $(\hat{\mu}, \hat{\gamma})$ and Cov $(\hat{\sigma}, \hat{\gamma})$, respectively; the value of Cov $(\hat{\mu}, \hat{\sigma})$ when $N=200$, $Q1=0.00$, $Q2=0.0$ should be −0.0010.

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The natural logarithm of the likelihood function is written down for the $m-r$ order statistics remaining after censoring the $n-m$ largest and the r smallest observations of a sample of size $n(0 \leq r < m \leq n)$ from a three-parameter lognormal population. Its first partial derivatives with respect to the parameters, when equated to zero, yield the likelihood equations, and the negatives of its second partial derivatives with respect to the parameters are the elements of the information matrix. Algebraic solution of the likelihood equations is impossible, so it is necessary to resort to iteration on an electronic computer. The iterative procedure proposed is applicable to special cases in which one or two of the parameters are known as well as to the most general case in which all three parameters are unknown. A modification of the procedure allows circumvention of a certain anomaly which sometimes occurs in maximum-likelihood estimation of the parameters of a three-parameter lognormal population from small samples. The information matrix is inverted to obtain the asymptotic variances and covariances of the local-maximum-likelihood estimators, which are tabulated for various values of the censoring proportions $q_1 = r/n$ from below and $q_2 = (n-m)/n$ from above. Results are reported of a Monte Carlo study conducted to check the validity of the asymptotic variances and covariances and their applicability to samples of moderate size.

1. INTRODUCTION

In their book on the lognormal distribution, Aitchison and Brown (1957, pp. 37–65) have devoted two entire chapters to estimation problems, one each for the two-parameter and three-parameter distributions. They have given a comprehensive summary of efforts up to that time to estimate the parameters of a lognormal population by the method of maximum likelihood, the method of moments, the method of quantiles, the graphical method, and mixed methods. The problem of estimating the parameters of a two-parameter lognormal population with known lower bound τ is equivalent to that of estimating the parameters of a normal population, which has been considered by a number of authors. Harter and Moore (1966) have summarized and extended the contributions of others to the solution of that problem, with particular emphasis on the method of maximum likelihood, and have proposed an iterative procedure for obtaining maximum-likelihood estimates of the parameters from complete, singly censored, and doubly censored samples.

Maximum-likelihood estimation of the parameters of a three-parameter lognormal population, for complete samples, has been investigated by Wilson and Worcester (1945), Cohen (1951), Aitchison and Brown (1957, pp. 55–6), and Hill (1963). The latter has explored some unusual features of the likelihood function of the three-parameter lognormal population which had apparently gone unnoticed by earlier investigators. In particular, he has shown that

there exist paths along which the likelihood function of any ordered sample $x_1, \cdots, x_n$ tends to ∞ as (τ, μ, σ) approaches $(x_1, -\infty, +\infty)$, where τ is the threshold parameter and μ and σ are the mean and the standard deviation of the parent normal population. This global maximum of the likelihood function leads to the ridiculous maximum-likelihood estimates $\hat{\tau}=x_1$, $\hat{\mu}=-\infty$, and $\hat{\sigma}=+\infty$ regardless of the sample. On the other hand, solution of the likelihood equations leads, in most cases, to local-maximum-likelihood estimates which, while not true maximum-likelihood estimates according to the usual definition, are reasonable estimates and appear to possess most of the desirable properties usually associated with maximum-likelihood estimates. Exceptions may occur in the case of small samples, for which the likelihood function may have no clearly defined local maximum.

Apparently nothing has been published on the problem of estimation for truncated or censored samples from a three-parameter lognormal population, though Aitchison and Brown (1957, pp. 88–91) have discussed the two-parameter case. The present paper will be devoted to local-maximum-likelihood estimation, for the three-parameter case, from singly and doubly censored as well as complete samples. An iterative estimation procedure for use on a high-speed computer will be given. This procedure will, in most cases, converge to a point where the likelihood function has a local maximum. If the sample (after censoring, if any) is small, convergence may be slow or the iterative procedure may take off along the path to infinity mentioned above. Even in the latter case, reasonable estimates can be obtained by a modification of the iterative procedure similar to that used by Harter and Moore (1965) for the three-parameter Gamma and Weibull populations.

2. THE LIKELIHOOD EQUATIONS

Consider a random sample of size n from a three-parameter lognormal population with parameters μ, σ, and τ (the location or threshold parameter). Let $X_{r+1}, \cdots, X_m$ be the ordered observations remaining after the $n-m$ largest and the r smallest observations have been censored. The joint probability density function of these order statistics is given by

$$\begin{aligned} f(x_{r+1}, \cdots, x_m; \mu, \sigma, \tau) &= \frac{n!}{(n-m)!r!} \prod_{i=r+1}^{m} \frac{1}{\sigma\sqrt{2\pi}(x_i - \tau)} \exp\left\{-\sum_{i=r+1}^{m} \frac{[\ln(x_i-\tau)-\mu]^2}{2\sigma^2}\right\} \\ &\quad \cdot \left\{1 - F\left[\frac{\ln(x_m-\tau)-\mu}{\sigma}\right]\right\}^{n-m} \left\{F\left[\frac{\ln(x_{r+1}-\tau)-\mu}{\sigma}\right]\right\}^{r}. \end{aligned} \tag{2.1}$$

The natural logarithm of the likelihood function is given by

$$\begin{aligned} L &= \ln[n!/(n-m)!r!] - \frac{1}{2}(m-r)\ln 2\pi - (m-r)\ln\sigma - \sum_{i=r+1}^{m}\ln(x_i-\tau) \\ &\quad - \frac{1}{2}\sum_{i=r+1}^{m} z_i^2 + (n-m)\ln[1-F(z_m)] + r\ln F(z_{r+1}). \end{aligned} \tag{2.2}$$

where

$$z_i = [\ln(x_i - \tau) - \mu]/\sigma,\ F(z_i) = \int_{-\infty}^{z_i} f(t)dt, \quad \text{and} \quad f(z_i) = (2\pi)^{-\frac{1}{2}} \exp(-z_i^2/2).$$

The likelihood equation are

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma}\left\{ \sum_{i=r+1}^{m} z_i + (n-m)\frac{f(z_m)}{1-F(z_m)} - r\frac{f(z_{r+1})}{F(z_{r+1})} \right\} = 0, \tag{2.3}$$

$$\frac{\partial L}{\partial \sigma} = \frac{1}{\sigma}\left\{ -(m-r) + \sum_{i=r+1}^{m} z_i^2 + (n-m)\frac{z_m f(z_m)}{1-F(z_m)} - r\frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})} \right\} = 0, \tag{2.4}$$

$$\frac{\partial L}{\partial \tau} = \sum_{i=r+1}^{m} (x_i - \tau)^{-1} + \frac{1}{\sigma}\left\{ \sum_{i=r+1}^{m} \frac{z_i}{x_i - \tau} + (n-m)\frac{f(z_m)}{(x_m - \tau)[1 - F(z_m)]} - r\frac{f(z_{r+1})}{(x_{r+1} - \tau)F(z_{r+1})} \right\} = 0. \tag{2.5}$$

If $m=n$ and $r=0$, equations (2.3) and (2.4) can be solved for μ and σ as explicit functions of τ, yielding

$$\hat{\mu} = \sum_{i=1}^{n} [\ln(x_i - \tau)]/n, \tag{2.6}$$

$$\hat{\sigma} = \sqrt{\sum_{i=1}^{n} [\ln(x_i - \tau) - \hat{\mu}]^2/n}.$$

The equation (2.5) cannot be solved explicitly even if $m=n$ and $r=0$. If censoring occurs, none of the likelihood equations has an explicit solution, and it is necessary to resort to iterative solutions. The details of an interative procedure for solving the likelihood equations will be given in section 4.

3. ASYMPTOTIC VARIANCES AND COVARIANCES OF ESTIMATORS

The second partial derivatives of L with respect to the parameters are

$$\frac{\partial^2 L}{\partial \mu^2} = \frac{1}{\sigma^2}\left\{ -(m-r) + (n-m)\frac{f(z_m)}{1-F(z_m)}\left[z_m - \frac{f(z_m)}{1-F(z_m)} \right] - r\frac{f(z_{r+1})}{F(z_{r+1})}\left[z_{r+1} + \frac{f(z_{r+1})}{F(z_{r+1})} \right] \right\}, \tag{3.1}$$

$$\frac{\partial^2 L}{\partial \sigma^2} = \frac{1}{\sigma^2}\left\{ (m-r) - 3\sum_{i=r+1}^{m} z_i^2 + (n-m)\frac{z_m f(z_m)}{1-F(z_m)}\left[z_m^2 - \frac{z_m f(z_m)}{1-F(z_m)} - 2 \right] - r\frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})}\left[z_{r+1}^2 + \frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})} - 2 \right] \right\}, \tag{3.2}$$

$$\frac{\partial^2 L}{\partial \tau^2} = \sum_{i=r+1}^{m} (x_i - \tau)^{-2} + \frac{1}{\sigma^2}\left\{ \sum_{i=r+1}^{m} \frac{\sigma z_i - 1}{(x_i - \tau)^2} + (n-m)\frac{f(z_m)}{(x_m - \tau)^2[1 - F(z_m)]} \right.$$

$$\cdot\left[z_m - \frac{f(z_m)}{1 - F(z_m)} + \sigma\right] \qquad (3.3)$$
$$- r\,\frac{f(z_{r+1})}{(x_{r+1} - \tau)^2 F(z_{r+1})}\left[z_{r+1} + \frac{f(z_{r+1})}{F(z_{r+1})} + \sigma\right]\Bigg\},$$

$$\frac{\partial^2 L}{\partial\mu\partial\sigma} = \frac{1}{\sigma^2}\Bigg\{-2\sum_{i=r+1}^{m} z_i + (n - m)\,\frac{f(z_m)}{1 - F(z_m)}\left[z_m^2 - \frac{z_m f(z_m)}{1 - F(z_m)} - 1\right] \qquad (3.4)$$
$$- r\,\frac{f(z_{r+1})}{F(z_{r+1})}\left[z_{r+1}^2 + \frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})} - 1\right]\Bigg\},$$

$$\frac{\partial^2 L}{\partial\mu\partial\tau} = \frac{1}{\sigma^2}\Bigg\{-\sum_{i=r+1}^{m}(x_i - \tau)^{-1} + (n - m)\,\frac{f(z_m)}{(x_m - \tau)[1 - F(z_m)]}\left[z_m - \frac{f(z_m)}{1 - F(z_m)}\right] \qquad (3.5)$$
$$- r\,\frac{f(z_{r+1})}{(x_{r+1} - \tau)F(z_{r+1})}\left[z_{r+1} + \frac{f(z_{r+1})}{F(z_{r+1})}\right]\Bigg\},$$

$$\frac{\partial^2 L}{\partial\sigma\partial\tau} = \frac{1}{\sigma^2}\Bigg\{-2\sum_{i=r+1}^{m}\frac{z_i}{(x_i - \tau)} + (n - m)\,\frac{f(z_m)}{(x_m - \tau)[1 - F(z_m)]}$$
$$\cdot\left[z_m^2 - \frac{z_m f(z_m)}{1 - F(z_m)} - 1\right] \qquad (3.6)$$
$$- r\,\frac{f(z_{r+1})}{(x_{r+1} - \tau)F(z_{r+1})}\left[z_{r+1}^2 + \frac{z_{r+1} f(z_{r+1})}{F(z_{r+1})} - 1\right]\Bigg\}.$$

Let

$$q_1 = r/n, \qquad q_2 = (n - m)/n, \quad \text{and} \quad p = 1 - q_1 - q_2 = (m - r)/n.$$

As $n \to \infty$ (q_1 and q_2 fixed), $z_{r+1} \to \hat{z}_{r+1}$ where

$$\int_{-\infty}^{\hat{z}_{r+1}} f(t)dt = q_1, \qquad z_m \to \hat{z}_m \quad \text{where} \quad \int_{\hat{z}_m}^{\infty} f(t)dt = q_2,$$

$$E\sum_{i=r+1}^{m} z_i \to \int_{\hat{z}_{r+1}}^{\hat{z}_m} tf(t)dt = -[f(\hat{z}_m) - f(\hat{z}_{r+1})],$$

$$E\sum_{i=r+1}^{m} z_i^2 \to \int_{\hat{z}_{r+1}}^{\hat{z}_m} t^2 f(t)dt = p - [\hat{z}_m f(\hat{z}_m) - \hat{z}_{r+1} f(\hat{z}_{r+1})],$$

$$E\sum_{i=r+1}^{m}\{(x_i - \tau)^{-1}\} \to \int_{\hat{z}_{r+1}}^{\hat{z}_m} e^{-(\mu+\sigma t)} f(t)dt$$
$$= e^{-\mu+\sigma^2/2}[F(\hat{z}_m + \sigma) - F(\hat{z}_{r+1} + \sigma)],$$

$$E\sum_{i=r+1}^{m}\{(x_i - \tau)^{-2}\} \to \int_{\hat{z}_{r+1}}^{\hat{z}_m} e^{-2(\mu+\sigma t)} f(t)dt$$
$$= e^{2(\sigma^2-\mu)}[F(\hat{z}_m + 2\sigma) - F(\hat{z}_{r+1} + 2\sigma)],$$

$$
\begin{aligned}
E\sum_{i=r+1}^{m}\{z_i/(x_i-\tau)\} &\to \int_{\hat z_{r+1}}^{\hat z_m} e^{-(\mu+\sigma t)} t f(t)\,dt \\
&= e^{-\mu+\sigma^2/2}\{\sigma[F(\hat z_m+\sigma)-F(\hat z_{r+1}+\sigma)] \\
&\quad + f(\hat z_m+\sigma)-f(\hat z_{r+1}+\sigma)\},
\end{aligned}
$$

$$
\begin{aligned}
E\sum_{i=r+1}^{m}\{(\sigma z_i-1)/(x_i-\tau)^2\} &\to \int_{\hat z_{r+1}}^{\hat z_m} e^{-2(\mu+\sigma t)}(\sigma t-1) f(t)\,dt \\
&= e^{2(\sigma^2-\mu)}\{(2\sigma^2+1)[F(\hat z_m+2\sigma) \\
&\quad - F(\hat z_{r+1}+2\sigma)] + \sigma[f(\hat z_m+2\sigma)-f(\hat z_{r+1}+2\sigma)]\}.
\end{aligned}
$$

The elements of the information matrix (multiplied by σ^2/n) may be written as

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\mu^2) &= p - f(\hat z_m)[\hat z_m - f(\hat z_m)/q_2] \\
&\quad + f(\hat z_{r+1})[\hat z_{r+1} + f(\hat z_{r+1})/q_1] = v^{11},
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\sigma^2) &= 2p - \hat z_m f(\hat z_m) - \hat z_m^2 f(\hat z_m)[\hat z_m - f(\hat z_m)/q_2] \\
&\quad + \hat z_{r+1} f(\hat z_{r+1}) + \hat z_{r+1}^2 f(\hat z_{r+1})[\hat z_{r+1} + f(\hat z_{r+1})/q_1] = v^{22},
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\tau^2) &= e^{2(\sigma^2-\mu)}\{(\sigma^2+1)[F(\hat z_m+2\sigma)-F(\hat z_{r+1}+2\sigma)] \\
&\quad + \sigma[f(\hat z_m+2\sigma)-f(\hat z_{r+1}+2\sigma)]\} \\
&\quad - e^{-2(\mu+\hat z_m\sigma)} f(\hat z_m)[\hat z_m - f(\hat z_m)/q_2 + \sigma] \\
&\quad + e^{-2(\mu+\hat z_{r+1}\sigma)} f(\hat z_{r+1})[\hat z_{r+1} + f(\hat z_{r+1})/q_1 + \sigma] = v^{33},
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\mu\partial\sigma) &= -f(\hat z_m) - \hat z_m f(\hat z_m)[\hat z_m - f(\hat z_m)/q_2] + f(\hat z_{r+1}) \\
&\quad + \hat z_{r+1} f(\hat z_{r+1})[\hat z_{r+1} + f(\hat z_{r+1})/q_1] = v^{12},
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\mu\partial\tau) &= e^{-\mu+\sigma^2/2}[F(\hat z_m+\sigma)-F(\hat z_{r+1}+\sigma)] \\
&\quad - e^{-(\mu+\hat z_m\sigma)} f(\hat z_m)[\hat z_m - f(\hat z_m)/q_2] \\
&\quad + e^{-(\mu+\hat z_{r+1}\sigma)} f(\hat z_{r+1})[\hat z_{r+1} + f(\hat z_{r+1})/q_1] = v^{13},
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(-\sigma^2/n)E(\partial^2 L/\partial\sigma\partial\tau) &= -2e^{-\mu+\sigma^2/2}\{\sigma[F(\hat z_m+\sigma)-F(\hat z_{r+1}+\sigma)] \\
&\quad + f(\hat z_m+\sigma)-f(\hat z_{r+1}+\sigma)\} \\
&\quad - e^{-(\mu+\hat z_m\sigma)} f(\hat z_m)[\hat z_m^2 - \hat z_m f(\hat z_m)/q_2 - 1] \\
&\quad + e^{-(\mu+\hat z_{r+1}\sigma)} f(\hat z_{r+1})[\hat z_{r+1}^2 - \hat z_{r+1} f(\hat z_{r+1})/q_1 - 1] = v^{23}.
\end{aligned} \tag{3.16}
$$

The asymptotic variance-covariance matrix for the estimators $\hat\mu$, $\hat\sigma$, and $\hat\tau$ is then $(\sigma^2/n)[v_{ij}]$, where $[v_{ij}]=[v^{ij}]^{-1}$. The reader will note that v^{11}, v^{22}, and v^{12} are independent of both μ and σ, while v^{33}, v^{13}, and v^{23} are dependent upon both. The dependence upon μ is only through the factor $e^{-\mu}$ in v^{13} and v^{23} and the

factor $e^{-2\mu}$ in v^{33}, while the dependence upon σ is more complicated. It follows that, among the elements of the inverse matrix $[v_{ij}]=[v^{ij}]^{-1}$, v_{11}, v_{22}, and v_{12} are independent of μ, v_{13} and v_{23} depend upon μ only through the factor e^{μ}, and v_{33} depends upon μ only through the factor $e^{2\mu}$.

The computation of the elements v^{ij} of the information matrix (multiplied by σ^2/n) and the inversion of this matrix and its submatrices and multiplication by σ^2 to obtain the coefficients of $1/n$ in the asymptotic variances and covariances were performed on the IBM 1620 computer. The resulting coefficients of $1/n$ in Var ($\hat{\mu}$), Var ($\hat{\sigma}$), Var ($\hat{\tau}$), Cov ($\hat{\mu}$, $\hat{\sigma}$), Cov ($\hat{\mu}$, $\hat{\tau}$), and Cov ($\hat{\sigma}$, $\hat{\tau}$) are given in Table 1 for $\mu=4$, $\sigma=2$ with censoring proportions $q_1=0.00$, 0.01, 0.02 and $q_2=0.0$, 0.5, together with the coefficients of $1/n$ in the variances and covariances when one parameter is known and in the variances when two parameters are known. For $\sigma=2$ and other values of μ, the coefficients can be obtained from the tabular values in Table 1 by multiplying by the proper exponential function of μ, but for other values of σ the computations must be made afresh. The coefficients tabulated are those of the local-maximum-

TABLE 1. COEFFICIENTS IN ASYMPTOTIC VARIANCES AND COVARIANCES OF MAXIMUM-LIKELIHOOD ESTIMATORS OF PARAMETERS μ, σ, AND τ OF LOGNORMAL POPULATION WITH $\sigma=2$ FROM SAMPLES OF SIZE N WITH PROPORTIONS Q1 CENSORED FROM BELOW AND Q2 FROM ABOVE

Q1	Q2	N VAR ($\hat{\mu}$)	N VAR ($\hat{\sigma}$)	N VAR ($\hat{\tau}$) / EXP ($2\mu-8$)	N COV ($\hat{\mu}$, $\hat{\sigma}$)	N COV ($\hat{\mu}$,$\hat{\tau}$) / EXP ($\mu-4$)	N COV ($\hat{\sigma}$,$\hat{\tau}$) / EXP ($\mu-4$)
0.00	0.0	4.015152	2.060608	0.827274	−0.030304	−10.111959	0.223919
0.00	0.5	6.079645	5.254124	0.850419	2.477933	0.097897	0.495163
0.01	0.0	4.214417	2.587046	19.696570	−10.352750	−2.036080	3.301319
0.01	0.5	6.120614	8.255096	25.148672	2.835131	1.023280	8.845586
0.02	0.0	4.332557	2.807065	38.013137	−10.313932	− 3.505939	5.304855
0.02	0.5	6.129487	9.808393	52.524255	2.899656	1.289503	15.355472

Q1	Q2	N VAR ($\hat{\mu}\mid\tau$)	N VAR ($\hat{\sigma}\mid\tau$)	N COV ($\hat{\mu}$,$\hat{\sigma}\mid\tau$)	N VAR ($\hat{\mu}\mid\sigma$)	N VAR ($\hat{\tau}\mid\sigma$) / EXP ($2\mu-8$)	N COV ($\hat{\mu}$,$\hat{\tau}\mid\sigma$) / EXP ($\mu-4$)
0.00	0.0	4.000000	2.000000	0.000000	4.014706	0.802941	−0.108666
0.00	0.5	6.068376	4.965812	2.420932	4.911011	0.803754	−0.135630
0.01	0.0	4.003943	2.033715	−0.011485	4.166319	15.483768	−1.585937
0.01	0.5	6.084977	5.143893	2.475215	5.152916	15.670572	−2.014615
0.02	0.0	4.009206	2.066755	−0.024667	4.238464	27.987902	−2.534700
0.02	0.5	6.097829	5.319219	2.522670	5.272261	28.484587	−3.250036

Q1	Q2	N VAR ($\hat{\sigma}\mid\mu$)	N VAR ($\hat{\tau}\mid\mu$) / EXP ($2\mu-8$)	N COV ($\hat{\sigma}$,$\hat{\tau}\mid\mu$) / EXP ($\mu-4$)	NVAR($\hat{\mu}\mid\sigma,\tau$)	N VAR($\hat{\sigma}\mid\mu,\tau$)	NVAR ($\hat{\tau}\mid\mu,\sigma$) / EXP ($2\mu-8$)
0.00	0.0	2.060379	0.824152	0.223074	4.000000	2.000000	0.800000
0.00	0.5	4.244172	0.848843	0.455262	4.888124	4.000000	0.800008
0.01	0.0	2.557520	18.712894	3.130898	4.003878	2.033683	14.880071
0.01	0.5	6.943120	24.977762	8.371956	4.893916	4.137038	14.882926
0.02	0.0	2.746102	35.176104	4.888977	4.008911	2.066603	26.472094
0.02	0.5	8.436662	52.252974	14.745451	4.901438	4.275592	26.481132

likelihood estimates resulting from the iterative estimation procedure which will be discussed in Section 4, not those of the global maxima. This fact is verified by the results of a Monte Carlo study, which will be reported in Section 5, comparing variances and covariances of estimates obtained from 500 random samples of each of various sizes and various degrees of censoring with those given by the asymptotic formulas.

4. ITERATIVE ESTIMATION PROCEDURE

The procedure for iterative estimation on a high-speed computer involves estimating the three parameters, one at a time, in the cyclic order μ, σ, τ, omitting any assumed to be known. Assuming that the first m order statistics of a sample of size $n(m \leq n)$ are known, except for the first $r_0(0 \leq r_0 \leq m-p$, where $p=1$, 2, or 3 is the number of parameters to be estimated), one starts by setting $r=r_0$. One then chooses initial estimates for the unknown parameters.

At each step, the rule of false position (iterative linear interpolation) is used to determine the value (if any) of the parameter then being estimated which satisfies the appropriate likelihood equation, in which the latest estimates (or known values) of the other two parameters have been substituted. For $\hat{\tau} < x_{r_0+1}$, one can always find estimates μ (finite) and σ (finite and positive) in this way. In estimating τ, however, one may find that no value of τ in the permissible interval $\tau \leq x_{r_0+1}$ satisfies the likelihood equation (2.5). In such cases, the likelihood function is monotone increasing, so that $\hat{\tau} = x_{r_0+1}$. As $\hat{\tau} \rightarrow x_{r_0+1}$, $\hat{\mu} \rightarrow -\infty$ and $\hat{\sigma} \rightarrow +\infty$, so that the estimation is proceeding along a path to the global maximum [see Hill (1963)]. When this occurs, which is not unusual when the available sample (after censoring, if any) is small, it is still possible to obtain reasonable estimates by a slight modification of the procedure. The modification entails censoring x_{r_0+1}, the smallest observation not previously censored, and any others equal to it, thus increasing r from r_0 to r_0+r_1, where $r_1 \geq 1$ is the number of observations censored at this point. Subsequently, x_{r_0+1} plays no role in the estimation procedure except as an upper bound on $\hat{\tau}$. Now the likelihood function is bounded and finite estimates $\hat{\mu}$ and $\hat{\sigma}$ are obtained. Iteration continues until the results of successive steps agree to within assigned tolerances (say 10^{-4}) or for a specified number of steps (say 550).

Use of the modified procedure calls for distributions of estimates which are conditional on the necessity for censoring the smallest previously uncensored observation(s). Lacking these, one may assert that the asymptotic variances of the estimators for the "sometimes censor" procedure are bounded below by those for the "never censor" procedure and above by those for the "always censor" procedure.

5. MONTE CARLO STUDY OF ESTIMATES

In order to check the validity of the asymptotic variances and covariances determined in Section 3 and their applicability to samples of moderate size, a Monte Carlo study was carried out on the IBM 7094 computer. Five hundred pseudo-random samples each of sizes 50, 100, and 200 from a lognormal pop-

ulation with parameters $\mu=4$, $\sigma=2$, and $\tau=10$ were generated in the computer. The iterative procedure described in Section 4 was used to estimate all three parameters, also every pair of parameters and every single parameter, the known values being substituted for the parameters not being estimated. In the case of samples of size 100, this was done not only for the complete samples but also for the samples with proportions $q_1=0.01$ censored from below and $q_2=0.5$ from above, both singly and in combination. The unmodified procedure converged to within the assigned tolerances of 10^{-4} in the specified 550 steps or fewer without exception for the uncensored and the singly censored samples. Among the doubly censored samples, however, there were 46 in which it was necessary to resort to the modified procedure (censoring the second order statistic) in order to estimate all three parameters simultaneously and 23 in which it was necessary to do so in order to estimate σ and τ simultaneously with μ known. More severe censoring or the use of smaller samples can lead to slower convergence of the iterative procedure as well as to the need to resort to the modified procedure for a greater proportion of cases.

The means, variances, and covariances of the estimates, based on 500 samples, are given in Table 2. For the doubly censored case, Table 2 gives two sets of results, one excluding and the other including the cases in which it was necessary to resort to the modified procedure. A comparison of the means in Table 2 with the population parameters $\mu=4$, $\sigma=2$, and $\tau=10$ and a comparison of the variances and covariances in Table 2 with the asymptotic variances and covariances obtained by dividing the coefficients in Table 1 by the sample size leads one to the following tentative conclusions:

(1) The estimator $\hat{\mu}$ from complete samples has a negative bias which is approximately proportional to n^{-1}, the reciprocal of the sample size. The bias of $\hat{\mu}$ appears to be unaffected by knowledge of σ, but it is small or non-existent if τ if known. The bias remains negative and increases in absolute value for moderate censoring, but may be positive in cases of severe censoring—see conclusion (4).

(2) The estimator $\hat{\sigma}$ has a positive bias when τ is unknown and a negative bias when τ is known. The magnitude of the bias is roughly proportional to n^{-1}, and appears to be unaffected by knowledge of μ. It is increased by censoring, especially from below.

(3) The estimator $\hat{\tau}$ has a positive bias which is closely proportional to n^{-1}. The magnitude of the bias appears to be unaffected by knowledge of μ and/or σ and by censoring from above, but it is markedly increased by censoring even a single observation from below.

(4) Cases in which it is frequently necessary to resort to the modified procedure are characterized by fairly large positive biases of all the estimators, especially $\hat{\sigma}$ and $\hat{\tau}$. The biases are only moderately increased, however, by including rather than excluding the instances in which resort to the modified procedure is necessary.

(5) The variance of $\hat{\tau}$ for samples of moderate size is much larger than the value given by the asymptotic formula (see Table 1). The excess over the asymptotic value, however, appears to be proportional to n^{-2}, whereas the asymptotic value itself is proportional to n^{-1}. Thus, for sufficiently large sam-

TABLE 2. MEANS, VARIANCES, AND COVARIANCES OF ESTIMATES OF PARAMETERS FROM 500 RANDOM SAMPLES, EACH OF SIZE N, WITH PROPORTIONS Q1 CENSORED FROM BELOW AND Q2 FROM ABOVE, DRAWN FROM LOGNORMAL POPULATION WITH PARAMETERS $\mu=4$, $\sigma=2$, AND $\tau=10$

N	Q1	Q2	MEAN $(\hat{\mu})$	MEAN $(\hat{\mu}\mid\sigma)$	MEAN $(\hat{\mu}\mid\tau)$	MEAN $(\hat{\mu}\mid\sigma,\tau)$
50	0.00	0.0	3.9172	3.9295	3.9890	3.9890
100	0.00	0.0	3.9624	3.9082	4.0009	4.0009
200	0.00	0.0	3.9303	3.9017	4.0080	4.0080
100	0.00	0.5	3.9838	3.9490	3.9608	3.9912
100	0.01	0.0	3.8928	3.9182	4.0011	4.0010
100	0.01	0.5	4.0565	3.8856	3.9602	3.9912
100	0.01*	6.5	4.0729	3.8856	3.9602	3.9912

N	Q1	Q2	MEAN $(\hat{\sigma})$	MEAN $(\hat{\sigma}\mid\mu)$	MEAN $(\hat{\sigma}\mid\tau)$	MEAN $(\hat{\sigma}\mid\mu,\tau)$
50	0.00	0.0	2.0758	2.0958	1.9870	1.9774
100	0.00	0.0	2.0450	2.0541	1.9810	1.9913
200	0.00	0.0	2.0129	2.0180	1.9828	1.9881
100	0.00	0.5	2.0943	2.1273	1.9383	1.9709
100	0.01	0.0	2.1795	2.1852	1.9806	1.9911
100	0.01	0.5	2.4740	2.4975	1.9361	1.9697
100	0.01*	0.5	2.5202	2.5224	1.9361	1.9697

N	Q1	Q2	MEAN $(\hat{\tau})$	MEAN $(\hat{\tau}\mid\mu)$	MEAN $(\hat{\tau}\mid\sigma)$	MEAN $(\hat{\tau}\mid\mu,\sigma)$
50	0.00	0.0	10.5885	10.6103	10.6109	10.6046
100	0.00	0.0	10.3209	10.3252	10.3160	10.3131
200	0.00	0.0	10.1490	10.1520	10.1501	10.1499
100	0.00	0.5	10.3114	10.3349	10.3107	10.3133
100	0.01	0.0	10.8035	10.7983	10.7262	10.7140
100	0.01	0.5	10.8608	10.8663	10.7319	10.7143
100	0.01*	0.5	10.8640	10.8664	10.7319	10.7143

N	Q1	Q2	VAR $(\hat{\mu})$	VAR $(\hat{\sigma})$	VAR $(\hat{\tau})$	COV $(\hat{\mu},\hat{\sigma})$	COV (μ,τ)	COV (σ,τ)
50	0.00	0.0	0.0869	0.0693	0.6132	−0.0119	−0.0087	0.0663
100	0.00	0.0	0.0416	0.0312	0.1733	−0.0032	−0.0015	0.0232
200	0.00	0.0	0.0211	0.0121	0.0477	−10.0010	−0.0016	0.0072
100	0.00	0.5	0.0600	0.0961	0.2109	0.0296	0.0146	0.0628
100	0.01	0.0	0.0428	0.0351	0.2861	−0.0047	0.0126	0.0064
100	0.01	0.5	0.0628	0.1244	0.3011	0.0375	0.0239	0.0199
100	0.01*	0.5	0.0634	0.1383	0.2877	0.0410	0.0240	0.0187

N	Q1	Q2	VAR $(\hat{\mu}\mid\tau)$	VAR $(\hat{\sigma}\mid\tau)$	COV $(\hat{\mu},\hat{\sigma}\mid\tau)$	VAR $(\hat{\mu}\mid\sigma)$	VAR $(\hat{\tau}\mid\sigma)$	COV $(\hat{\mu},\hat{\tau}\mid\sigma)$
50	0.00	0.0	0.0798	0.0408	0.0006	0.0837	0.5240	0.0059
100	0.00	0.0	0.0406	0.0218	0.0002	0.0413	0.1536	0.0015
200	0.00	0.0	0.0208	0.0096	−0.0001	0.0211	0.0433	−0.0010
100	0.00	0.5	0.0570	0.0460	0.0198	0.0511	0.1540	−0.0009
100	0.01	0.0	0.0406	0.0224	0.0000	0.0422	0.2871	0.0104
100	0.01	0.5	0.0574	0.0473	0.0197	0.0518	0.2854	0.0130
100	0.01*	0.5	0.0574	0.0473	0.0197	0.0518	0.2854	0.0130

N	Q1	Q2	VAR $(\hat{\sigma}\mid\mu)$	VAR $(\hat{\tau}\mid\mu)$	COV $(\hat{\sigma},\hat{\tau}\mid\mu)$	VAR $(\hat{\mu}\mid\sigma,\tau)$	VAR $(\hat{\sigma}\mid\mu,\tau)$	VAR $(\hat{\tau}\mid\mu,\sigma)$
50	0.00	0.0	0.0692	0.6047	0.0603	0.0798	0.0407	0.5315
100	0.00	0.0	0.0308	0.1725	0.0221	0.0406	0.0216	0.1552
200	0.00	0.5	0.0120	0.0477	0.0069	0.0208	0.0096	0.0436
100	0.00	0.0	0.0789	0.1905	0.0469	0.0492	0.0393	0.1553
100	0.01	0.0	0.0349	0.2899	0.0070	0.0406	0.0222	0.2923
100	0.01	0.5	0.1162	0.2918	0.0066	0.0494	0.0407	0.2923
100	0.01*	0.5	0.1263	0.2859	0.0057	0.0494	0.0407	0.2923

* Results given on this line are for all 500 cases, including those, omitted on the line above, which required use of the modified procedure (censoring the second order statistic), so that Q1 = 0.02.

ples, the excess would become negligible in comparison with the asymptotic value. The excess appears to be decreased little if any by knowledge of μ and only slightly by knowledge of σ. The excess appears to be affected very little by censoring, but for censoring of even a single observation from below the asymptotic value increases markedly, so that the excess becomes smaller by comparison.

(6) When τ is known, the variances of $\hat{\mu}$ and $\hat{\sigma}$ are in close agreement with the values given by the asymptotic formulas. When τ is unknown, the variances of $\hat{\mu}$ and $\hat{\sigma}$ and their covariances with $\hat{\tau}$ are somewhat larger than the values given by the asymptotic formulas. The excess over the asymptotic value is greater for $\hat{\sigma}$ than for $\hat{\mu}$, as one would expect from the fact that $\hat{\sigma}$ is more strongly correlated with $\hat{\tau}$ than is $\hat{\mu}$.

(7) For cases in which it is frequently necessary to resort to the modified procedure, the variances and covariances including and excluding instances in which the modified procedure is actually used do not appear to differ systematically from each other or from those given by the asymptotic formulas, except that Cov $(\hat{\sigma}, \hat{\tau})$ is unexpectedly small.

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REFERENCES

Aitchison, J. and Brown, J. A. C. (1957). *The Lognormal Distribution*. Cambridge University Press, Cambridge, England.

Cohen, A. C. Jr. (1951). "Estimating Parameters of Logarithmic-Normal Distributions by Maximum Likelihood," *Journal of the American Statistical Association*, **46**, 206–12.

Harter, H. Leon and Moore, Albert H. (1965). "Maximum-Likelihood Estimation of the Parameters of Gamma and Weibull Populations from Complete and from Censored Samples," *Technometrics*, **7**, 639–43.

Harter, H. Leon and Moore, Albert H. (1966). "Iterative Maximum-Likelihood Estimation of the Parameters of Normal Populations from Singly and Doubly Censored Samples," *Biometrika*, **53**, 205–13.

Hill, Bruce M. (1963). "The Three-Parameter Lognormal Distribution and Bayesian Analysis of a Point-Source Epidemic," *Journal of the American Statistical Association*, **58**, 72–84.

Wilson, Edwin B. and Worcester, Jane (1945). "The Normal Logarithmic Transform," *Review of Economic Statistics*, **27**, 17–22.